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Killing–Yano Tensors in General Relativity

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It is shown that space-times admitting more than one independent Killing-Yano tensor belong to a small collection of highly idealised space-times. A new characterization of Robertson-Walker space-times arises as a corollary of the main theorem.

1. INTRODUCTION

There has been some recent interest in Killing-Yano (KY) tensors in general relativity. The purpose of the present paper is to establish some further general results concerning the existence of KY tensors and to show that space-times admitting more than one independent KY tensor belong to a small collection of highly specialized space-times. Only local problems will be considered here.

Within a connected coordinate domain U of a space-time a KY tensor is taken as a smooth bivector F that is nowhere zero in U and satisfies

$$F_{ab;c}+F_{ac;b}=0 \tag{1}$$

where a semicolon denotes a covariant derivative with respect to the space-time Lorentz metric g_{ab} on U (and the latter is taken with signature $(-,+,+,+)$). A KY tensor is called *simple* if it is a simple bivector at each point of U. Such a KY tensor will then be called *timelike* (respectively *spacelike*, *null*) if its blade is everywhere timelike (respectively spacelike, null). A KY tensor is called *nonsimple* if it is not simple at any point of U.[2] If a KY tensor is not a simple bivector at $p\in U$, then it uniquely determines a canonical pair of orthogonal blades at p, one timelike (and spanned by the two distinct null eigendirections of the bivector at p) and one spacelike. In this sense, nonnull, simple KY tensors also determine a

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[2]The terms simple and nonsimple may be taken as opposites in the local sense considered here because if a bivector is nonsimple at $p\in U$ it remains so in some neighborhood of p which may be then taken as the set U.

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canonical pair of orthogonal blades at each $p \in U$, its actual blade and its orthogonal complement.

2. GENERAL RESULTS ON KILLING–YANO TENSORS

The following nine results will now be established concerning KY tensors on U. Many are believed to be new.

(i) There are at most ten independent solutions to equation (1) and this maximum occurs if and only if the domain U has constant curvature.

(ii) If a null KY tensor F is admitted, the Weyl tensor is at each $p \in U$ either zero or of Petrov type N with the unique null direction in the blade of F as repeated principal null direction. If a nonnull KY tensor F is admitted, the Weyl tensor is at each $p \in U$ either zero or of Petrov type D with its two repeated principal null directions spanning the blade of F if it is timelike, the 2-space orthogonal to the blade of F if it is spacelike, and the timelike member of its canonical pair of blades if it is nonsimple.

(iii) If a simple KY tensor is admitted, then each member of its blade is a Ricci eigenvector with the same eigenvalue at each $p \in U$.

(iv) If F is a KY tensor that is not simple at p, then each member of the timelike canonical blade at p is a Ricci eigenvector with the same eigenvalue. Similar comments apply to the spacelike canonical blade at p (but the two resulting eigenvalues need not be equal).

(v) If a simple KY tensor F is admitted, the natural map that associates $p \in U$ with the blade of F at p is an integrable (surface-forming) two-dimensional distribution (in the sense of Frobenius) on U.

(vi) If no point of U has constant curvature and two independent KY tensors are admitted, one of which is nonsimple, then no further independent KY tensors are admitted. Also, U admits two independent recurrent null vector fields l and n. The two KY tensors determine the same pair of canonical blades at each $p \in U$ with the timelike one containing the recurrent null vector fields. By taking appropriate linear combinations of these KY tensors, one can take them as two covariantly constant, orthogonal, simple bivectors whose blades are the same as the canonical pair of blades of the original KY tensors. At each $p \in U$, the Petrov type is O or D with l and n spanning the repeated principal null directions and the Ricci tensor is of Segré type $\{(1,1)(1\ 1)\}$ or its degeneracy with eigenvalue degeneracies on the above canonical pair of blades. The Ricci tensor is not zero at any $p \in U$.

(vii) If no point of U has constant curvature and a null KY tensor F is admitted, the repeated principal null direction l of F may be scaled so that it is a null Killing vector field and gives rise to a null, shear-free, expansion-free, twist-free geodesic congruence in U. At each $p \in U$, the Petrov type is O or N with l spanning the repeated principal null direction.

If the Ricci tensor is zero at each $p \in U$, then the dual $\overset{*}{F}$ of F is also a KY tensor (and no other independent KY tensors are admitted) and U is (part of) a *pp*-wave space-time.

(viii) If no point of U has constant curvature and two independent null KY tensors with the same repeated principal null direction l are admitted, they may be taken as a dual pair of covariantly constant null KY tensors and l may be scaled to a constant null vector field l^a on U. No further KY tensors are admitted and the Petrov type is either O or N with l as repeated principal null direction. The Ricci tensor is either zero or proportional to $l_a l_b$.

(ix) If no point of U has constant curvature and if two independent, simple, nonnull KY tensors with intersecting blades are admitted, then, if at each $p \in U$ these blades span a spacelike (timelike) 3-space, the metric on U is of the Robertson–Walker type ("spacelike" Robertson–Walker type with "fluid" flow vector spacelike). It is not possible for them to span a null 3-space at each $p \in U$.

Proof:

(i) This follows by noting that the proof of this result in the positive-definite case (Tachibana, 1968) holds in the Lorentz case.

(ii) This was given for vacuum space-times in Collinson (1974) and for the general case in Stephani (1978).

(iii, iv) The integrability conditions for (1) impose the restrictions

$$R_{ac}F^c_{\ b} + R_{bc}F^c_{\ a} = 0 \tag{2}$$

on the Ricci tensor components R_{ab} (Stephani, 1978). The results (iii) and (iv) now follow from the general algebraic study of equation (2) given in Hall and McIntosh (1983). [Details of the algebraic Segré-type structure of the Ricci tensor used later in the paper can be found in Hall (1976, 1984).]

(v) Because F is smooth and nowhere zero on U, the map that associates each $p \in U$ with the blade of F can be shown to be a two-dimensional smooth distribution on U. Then one may write $F_{ab} = 2r_{[a}s_{b]}$, where r and s are nowhere zero, smooth one-forms on U, and where the square brackets denote skew-symmetrization. A substitution into (1) and a contraction with g^{ac} gives the desired result.

(vi) Let F be a nonsimple KY tensor in U. The canonical blade structure of F determines two two-dimensional distributions on U and so one may choose a real tetrad (l, n, x, y) of smooth vector fields on U such that $l^a n_a = x^a x_a = y^a y_a = 1$ and all other inner products between tetrad members are zero at each $p \in U$ and

$$F_{ab} = 2Al_{[a}n_{b]} + 2Bx_{[a}y_{b]} \tag{3}$$

where A and B are smooth, nowhere zero, real-valued functions on U. It is now clear from (ii)–(iv) that the Petrov type is O or D and that the Ricci tensor has Segré type $\{(1,1)(1,1)\}$ or its degeneracy, this degeneracy occurring if and only if the Ricci tensor admits eigenvectors other than those in the canonical pair of blades. As a consequence, any other independent KY tensor must be nonnull and, whether simple or nonsimple, must determine the same canonical pair of blades as F at each $p \in U$. Otherwise one could find a nonempty, open subset of U where the Weyl tensor vanished and where $R_{ab} \propto g_{ab}$, that is, where one had constant curvature, contradicting the initial assumption. So the other independent KY tensor G must take the general form (3) with A and B replaced by, say, α and β and where now G may be simple over a subset, possibly the whole of U. Also, since F is a KY tensor, if ϕ is a smooth, nowhere zero, real-valued function on U, then ϕF is a KY tensor if and only if ϕ is constant on U. As a consequence, the independence of F and G over U shows that the matrix

$$\begin{pmatrix} A & B \\ \alpha & \beta \end{pmatrix}$$

is never singular over an open subset of U. Equations (1) and (3) and some appropriate contractions with tetrad vectors show (Stephani, 1978) that l and n are shear-free, null, geodesic congruences and that

$$A_{,a}l^a = A_{,a}n^a = B_{,a}x^a = B_{,a}y^a = 0$$

where a comma denotes a partial derivative. One also finds that $\mathrm{Re}[(A+iB)(\theta+i\omega)]=0$, where θ and ω denote, respectively, the expansion and twist of the congruence l. A similar expression involving α and β can be obtained from the KY tensor G and the independence of F and G then shows that $\theta=\omega=0$ in U. Similar comments hold for the congruence n, and so l and n are each expansion-free, twist-free, shear-free, null, geodesic congruences.

Two further linear relationships, this time involving not θ and ω, but the real and imaginary parts of the rotation τ of the congruence l and the analogous quantity π for n, can be found and they lead to $\tau=\pi=0$. Thus, l and n are recurrent null vector fields, $l_{a;b}=l_a p_b$, $n_{a;b}=-n_a p_b$ for some smooth one-form p on P. Then $2l_{[a}n_{b]}$ and its dual $2x_{[a}y_{b]}$ are easily found to be covariantly constant bivectors. Using this information in (3) and substituting into (1), a contraction with l^a shows that A is constant, while a contraction with x^a shows that B is constant. The rest of the proof of (vi) now follows except for the last sentence. This can be established by a direct computation of the curvature tensor or, more quickly, by contradiction, since a vacuum space-time admitting a recurrent null vector field is of Petrov type III or N, as follows from the Ricci identity.

(vii) The results here are essentially known (Collinson, 1974; Stephani, 1978). One writes $F_{ab}=2l_{[a}x_{b]}$, where $l^a l_a = l^a x_a = 0$ in U and substitutes into (1). Appropriate contractions show that l is a shear-free, expansion-free, twist-free null geodesic congruence and hence may be scaled so that $l_{[a;b]}=0$. The result of contracting (1) first with l^a and then with x^a is that

$$2l_{a;b}=\psi_{,a}l_b+l_a\psi_{,b}, \qquad \psi=-\tfrac{1}{2}\ln(x^a x_a) \tag{4}$$

and so $l'^a \equiv e^{-\psi}l^a$ is a Killing vector field in U. If the vacuum condition holds, then the Riemann tensor R_{abcd} is of Petrov type N and so $R_{abcd}l'^d=0$. This condition together with the integrability condition on the Killing vector field l'^a gives $l'_{a;bc}=0$ and so l'^a is either a constant vector field or else $l'_{a;b}$ is a constant bivector field on U. Each of these alternatives leads to a pp-wave metric on U and to the existence of a constant null bivector H on U whose repeated principal null direction is l (Ehlers and Kundt, 1962). It now easily follows that H and $\overset{*}{H}$ are independent KY tensors on U and that any other KY tensor on U is of the form $\alpha H+\beta\overset{*}{H}$ with α and β constants.

(viii) Suppose $2l_{[a}r_{b]}$ and $2l_{[a}s_{b]}$ are null KY tensors with r and s spacelike and orthogonal to l. On substituting into (1) and contracting the two resulting equations with l^a, one obtains $l_{a;b}r^a=l_{a;b}s^a=0$. As a consequence, l is recurrent, $l_{a;b}=l_a p_b$ for some one-form p on U. To prove that l can be scaled to a constant null vector field on U, one notes from (vii) that it may be scaled to a Killing vector field on U and then uses the obvious fact that a recurrent Killing vector field is necessarily constant. Next, result (iii) shows that at each $p\in U$, every member of the null three-space at p spanned by l, r, and s is a Ricci eigenvector with the same eigenvalue. A simple study of the permissible Segré types for R_{ab} (see, for example, Hall, 1976, 1984) then shows that the trace-free part, $\tilde{R}_{ab}$, of R_{ab} is either zero or proportional to $l_a l_b$ and hence $E_{abcd}l^d=0$, where E is the anti-self-dual part of the Riemann tensor,

$$R_{abcd}=C_{abcd}+E_{abcd}+\tfrac{1}{6}RG_{abcd} \tag{5}$$

where

$$\begin{aligned} E_{abcd}&=\tilde{R}_{a[c}g_{d]b}+\tilde{R}_{b[d}g_{c]a}, \qquad & G_{abcd}&=g_{a[c}g_{d]b}\\ \tilde{R}_{ab}&=R_{ab}-\tfrac{1}{4}Rg_{ab}=E^{c}_{\ acb}, \qquad & R&=R_{ab}g^{ab}\end{aligned} \tag{6}$$

and where the C_{abcd} are the components of the Weyl tensor. Now result (ii) shows that either C_{abcd} is zero or else satisfies the type N condition $C_{abcd}l^d=0$, and the fact that l^a is covariantly constant, together with the Ricci identity, gives $R_{abcd}l^d=0$. A contraction of (5) with l^d then shows that the Ricci scalar $R=0$. Further, the existence of a dual pair of covariantly constant null bivectors with repeated principal null direction l now follows, since any null bivector F satisfying

$$F_{ab}l^b = \overset{*}{F}_{ab}l^b = 0$$

necessarily satisfies

$$\overset{+}{F}_{ab;c} = \overset{+}{F}_{ab}q_c \qquad (\overset{+}{F}_{ab} = F_{ab} + i\overset{*}{F}_{ab})$$

for some complex one-form on U and

$$\overset{+}{F}_{ab;[cd]} = 0.$$

As a consequence, q_a is locally the gradient of a complex-valued function ϕ on U, $q_a = \phi_{,a}$, and so

$$(e^{-\phi}\overset{+}{F}_{ab})_{;c} = 0$$

Finally, to show that no other independent KY tensors are admitted, note that another null KY tensor with a repeated principal null direction k distinct from l cannot be admitted because (ii) and (iii) then show that $C_{abcd} = 0$ and $R_{ab} = 0$, the latter result following because the Ricci eigenvector k must have the same eigenvalue as l, namely zero, due to the fact that $l_ak^a \neq 0$. The Ricci tensor then has four independent eigenvectors with zero eigenvalue and is thus zero and the assumption about constant curvature is contradicted. Similar arguments rule out the existence of a timelike KY tensor. If another null KY tensor with repeated principal null direction l is admitted, then it is easy to show that it must be a (constant) linear combination of the two covariantly constant null bivectors above and is therefore not independent. Consequently, if another independent KY tensor is admitted, then it must be spacelike, because part (vi) rules out the possibility of it being nonsimple. Further, its blade must be orthogonal to l (otherwise an argument given above would yield the contradiction $R_{ab} = 0$) and the Weyl tensor is zero. So choose coordinates (u, v, x, y) such that the metric becomes [due to the covariantly constant null bivector admitted and the conformal flatness (Kramer et al., 1980; see also Ehlers and Kundt, 1962)]

$$ds^2 = dx^2 + dy^2 + 2\,du\,dv + \phi(u)(x^2 + y^2)\,du^2 \tag{7}$$

where, if $l_a = u_{,a}$, $x_a = x_{,a}$, $y_a = y_{,a}$, the bivectors $2l_{[a}x_{b]}$ and $2l_{[a}y_{b]}$ are covariantly constant. Any other possible independent KY tensor H must take the form

$$H_{ab} = 2\alpha l_{[a}x_{b]} + 2\beta l_{[a}y_{b]} + 2\gamma x_{[a}y_{b]} \tag{8}$$

On substituting (8) into (1) and contracting with the tetrad vectors, one easily finds, first, that α and β are independent of v, second, that $\gamma = 0$, and, finally, that α and β are constants. Thus, H is not independent and the result follows.

(ix) Suppose F and G are independent nonnull, simple KY tensors whose blades intersect and span a null 3-space at each $p \in U$. Clearly, F and G must both be spacelike. Write $F_{ab} = 2\alpha x_{[a}y_{b]}$ with $x^a y_a = 0$, $x^a x_a = y^a y_a = 1$, and α a real-valued, nowhere-zero function on U, and let l be a null vector field on U that lies in the above null 3-space at each $p \in U$, so that $l^a x_a = l^a y_a = 0$. Choose a real null tetrad (l, n, x, y) on U. Now (ii) and (iii) show that the Weyl tensor $C = 0$ and that $R_{ab} = \lambda l_a l_b + \frac{1}{4} R g_{ab}$ (where λ is a nowhere-zero, real-valued function on U, otherwise the constant-curvature assumption would be contradicted). By substituting the above expression for F_{ab} into (1) and contracting with tetrad members (see, for example, Stephani, 1978), one finds that l is a shear-free, twist-free, rotation-free null geodesic congruence (and now l will be assumed affinely parametrized). The rotation-free condition $l_{a;b} m^a n^b = 0$, where $\sqrt{2} m^a = x^a + i y^a$, must also hold for a similar null tetrad built from the blade of G just as the present one was built from the blade of F. Since these blades are connected by a proper null rotation, one easily finds that the expansion of l also vanishes. As a consequence, l is recurrent and when this is used in the Ricci identity, together with the conditions $C = 0$ and $E_{abcd} l^d = 0$ (the latter following from the above-obtained canonical form for R_{ab}), an obvious contraction gives $R = 0$ and hence $R_{abcd} l^d = 0$. The recurrence vector for l is thus locally a gradient and l is scalable to a constant null vector. A contradiction now follows from the proof of part (viii), since one now has the conditions under which only null KY tensors are admitted.

Suppose now that F and G are as above, but where their blades intersect and span a spacelike 3-space at each $p \in U$. Clearly, F and G must be spacelike and one can write

$$F_{ab} = 2\alpha x_{[a} y_{b]}, \qquad G_{ab} = 2\beta y_{[a} z_{b]} \tag{9}$$

where x, y, and z constitute an orthogonal triad of spacelike vectors and α and β are real-valued, nowhere-zero functions on U. Let u be a timelike vector field orthogonal to x, y, and z at each $p \in U$. It is easily shown that u may be chosen to be smooth if x, y, and z are smooth. On substituting (9) into (1) and contracting with $u^a u^c$, one finds that $u^a_{;b} u^b$ is orthogonal to x, y, and z and hence u is geodesic. Also, parts (ii) and (iii) show that $C = 0$ and that x, y, and z are Ricci eigenvectors with equal eigenvalues and so $R_{ab} = \gamma u_a u_b + \delta g_{ab}$ for real-valued functions γ and δ on U. The conformally flat Bianchi identity can now be used to show that u is hypersurface orthogonal and hence that x, y, and z are hypersurface-forming and that γ and δ are constant on these hypersurfaces. Thus U is (part of) a Robertson–Walker space-time. To avoid contradicting the constant-curvature assumption, one must have γ nowhere zero on U, that is, for energy-momentum tensors that arise from perfect fluids, $p + \rho \neq 0$, where p

and ρ are, respectively, the usual pressure and density of the perfect fluid involved.

Similar comments apply when the (nonnull) blades of F and G span a timelike 3-space. In this case u is a spacelike, hypersurface-orthogonal geodesic vector field and the "spacelike" version of the Robertson-Walker metric is obtained.

From our results one can construct the following theorem governing the types of space-times that admit at least two independent KY tensors. It should be pointed out again that the conditions placed on the KY tensors in the previous results were assumed to hold throughout U and it is with this assumption understood that the theorem is stated.

Theorem. If the region U admits at least two independent KY tensors, then either

(a) it is of constant curvature and so admits ten independent KY tensors, or
(b) it is (part of) a Robertson-Walker space-time (or its "spacelike" equivalent) and is not an Einstein space and admits exactly four independent KY tensors, or
(c) it is a decomposable space-time of the type given in (vi) and admits exactly two independent KY tensors, or
(d) it is (part of) a (necessarily type O or N) space-time admitting a covariantly constant null bivector (and hence a covariantly constant null vector) and admits exactly two independent KY tensors.

If, further, U is a nonflat vacuum region, it is (part of) a *pp*-wave space-time and admits exactly two independent KY tensors.

Proof. The idea of the proof is to consider all the possibilities for two independent KY fields and then to use results (i)–(iv) to achieve either the condition (a) of the theorem (that is, to show that $C=0$ and that $R_{ab} \propto g_{ab}$) or one of the cases discussed in results (vi), (viii), or (ix). If one of the KY tensors is nonsimple, then the result (vi) shows that either (a) or (c), results so one can assume from now on that, at least, two independent simple KY F and G are admitted. If F and G are nonnull and determine the same canonical pairs of blades at each $p \in U$, so that one is timelike and one spacelike, then again either (a) or (c) results. If, on the other hand, their canonical blade pairs are different, then either their blades intersect [and so result (ix) shows that (a) or (b) results] or their blades intersect trivially in only the zero vector at each $p \in U$ (and are not orthogonal), in which case $C=0$ and all members of both blades are Ricci eigenvectors with the same eigenvalue, $R_{ab} \propto g_{ab}$, and so (a) results.

Now suppose that F is nonnull and G is null, so that, immediately, one has $C=0$ from result (ii). If the blades of F and G intersect only trivially, then, as above, one obtains (a). If their blades intersect, then the argument given in (ix) is easily adapted to show that (a) or (b) results unless F is spacelike, G null, and their blades span a null 3-space at each $p \in U$. In this case the first part of the proof of (ix) applied to F and result (vii) applied to G shows that the repeated principal null direction l of G can be scaled so that $l_{a;b}=0$ and that a constant null bivector is admitted. Since this constant null bivector and its dual span all the independent KY tensors if (a) does not hold, one is forced here to condition (a).

Next, suppose that F and G are both null. If their blades do not intersect, then results (ii) and (iii) show that $C=0$ and that $R_{ab} \propto g_{ab}$ and hence (a) results. If their blades intersect, but their repeated principal null directions are distinct so that the blades span a timelike 3-space at each $p \in U$, then again $C=0$ and a slight modification of the argument in the proof of result (ix) shows that (a) or (b) results. Finally, if their repeated principal null directions are the same, result (viii) shows that (d) holds.

The statements regarding the exact numbers of independent KY tensors admitted in (a), (c), and (d) follow from results (i), (vi), and (viii). That there are exactly four when condition (b) holds can be shown, for example, in the Robertson–Walker case by constructing the four independent KY tensors that must be admitted in the intrinsic geometry of the submanifolds of constant curvature given in the usual coordinates by $t=\text{const}$ and extending them to KY tensors in U. [That there are exactly four in a three-dimensional space of constant curvature follows from Tachibana (1968).] Writing the Robertson–Walker metric in a standard chart domain which will be identified with U as

$$ds^2 = -dt^2 + f^2 \gamma_{\alpha\beta}\, dx^\alpha\, dx^\beta \tag{10}$$

where Greek indices take the values 1, 2, and 3, where γ is a 3-space metric of constant curvature and f^2 is a positive function of t only, let $G_{\alpha\beta}$ be a KY tensor in a particular hypersurface S given by $t=t_0=\text{const}$ with the metric γ. Now construct a bivector F in U as follows: let $p \in U$ and choose F_{ab} at p to satisfy $F_{ao}(p)=0$, $F_{\alpha\beta}(p)=f^3(p)G_{\alpha\beta}(q)$, where q is the unique point of S where the path $x^\alpha=\text{const}$ from p cuts S. If G is smooth on S, then, since the associated projection $U \to S$ is smooth, F is smooth on U. It is easily checked that F is a KY tensor on U. The four independent KY tensors so generated in U are necessarily spacelike, since each must satisfy $F_{ab}u^b=0$. Further, no other independent KY tensors can be admitted in U because if another one G was admitted, then clearly $G_{ab}u^b=0$ must hold due to results (iii) and (iv) and so G gives rise, in an obvious way, to an intrinsic KY tensor in each of the space sections $t=\text{const}$. Because there are exactly four independent KY tensors in the intrinsic geometry of each $t=\text{const}$ hypersurface, it follows that G may be written in terms of the four independent KY tensors F_i in U ($i=1,2,3,4$), constructed as above from the four independent intrinsic ones in S, that is, $G=\sum_{i=1}^{4} \alpha_i F_i$, where each α_i depends only on t. On substituting this expression into (1) and contracting with u^c, one obtains $\sum \dot{\alpha}_i F_i=0$, where a dot denotes differentiation with respect to t. On restricting this equation to each of the hypersurfaces $t=\text{const}$, one finds that $\dot{\alpha}_i=0$ for each i and so each α_i is constant on U. This contradicts the independence of G and completes the proof.

3. DISCUSSION

A number of corollaries follow from the theorem of the last section. For example, it follows that the total number of independent KY tensors in U is either none, one, two, four, or ten (none, one, two, or ten *in vacuo*). If the energy-momentum tensor in U is that of a Maxwell field, then in the total number of independent KY tensors admitted is either none, one, or two and, in the last case, the null and nonnull Maxwell fields come under conditions (d) and (c) of the theorem, respectively. [It is clear from results (ii) and (iii) of the last section and the algebraic structure of the associated energy-momentum tensor that if a null (nonnull) Maxwell field admits a KY tensor then the latter must be null (nonnull); cf. Van Leeuwen (1981).] If the energy-momentum tensor in U is that of a perfect fluid where the pressure and density satisfy $p+e\neq 0$, the number of independent KY tensors admitted is either none, one, or four and they are necessarily spacelike. It is also perhaps of interest to note how result (ix) of the last section provides a characterization of Robertson–Walker metrics. This is brought out in the following corollary.

Corollary. If no point of U has constant curvature and at least two independent KY tensors are admitted, then either case (b) of the theorem holds, in which case exactly four independent KY tensors are admitted and no KY tensor is covariantly constant, or exactly two independent KY tensors are admitted and all KY tensors are covariantly constant.

To prove this, it is required only to show that no KY tensor in case (b) is covariantly constant. This follows easily because if a (necessarily spacelike) covariantly constant one is admitted, its (timelike) dual is also covariantly constant and hence a KY tensor and result (iii) in Section 2 complete the contradition, since one would obtain $R_{ab}\propto g_{ab}$ and constant curvature in U.

It is also noted that the "spacelike" Robertson–Walker metrics in condition (b) of the theorem can be eliminated by imposing the usual "dominant energy conditions" on U (see, for example, Kramer *et al.*, 1980).

It should be pointed out that there are several errors in a recent paper dealing with KY tensors (Taxiarchis, 1985). In this reference there is some confusion over the conformally flat case and the generality of the energy-momentum tensors discussed. Also, there is an apparent error in the final section of this reference, where it is claimed that the existence of a single null KY tensor implies that its (expansion-free, twist-free, shear-free, geodesic) repeated, principal null direction can be scaled so that its rotation is zero. I cannot see how this can be done and feel that results (vii) and (viii) above summarize the correct solution.

My attention has also recently been drawn to two further references (Dietz and Rüdiger, 1981, 1982), which give a very general discussion of KY tensors. However, these references also contain some confusing statements concerning the Petrov types.

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